Potential Energy & Conservation of Energy

Conservative and non-conservative forces

Consider an object that moves from an initial position $\vec{x}_f$ to a final position $\vec{x}_f$ under the influence of a force $\vec{F}$.

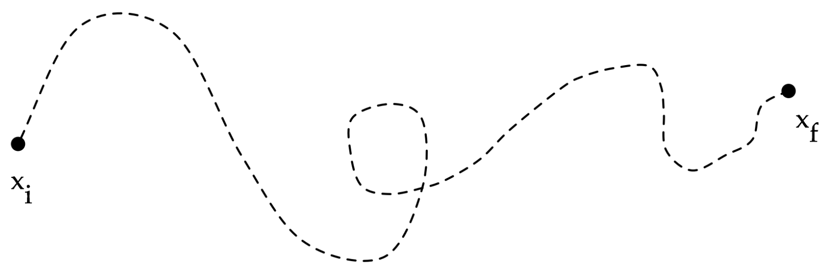

Figure 1: A particle moves from $\vec{x}_i$ to $\vec{x}_f$ along an arbitrary path.

The force is *conservative* if the work done by it on the object depends on the path traveled by the object[1]. Some examples of conservative forces that you will encounter in this course are the gravitational force and the force exerted by a spring. On the other hand, if the work done by the force depends on the shape, length, or any other aspect of the path taken by the object, then we label the force *non-conservative*. An example of such a force is the force of kinetic friction.

[1] An alternate definition: $\vec{F}$ is conservative if the work done for any closed path is zero (A closed path is one in which the object ends up back where it started, that is, its initial position is the same as its final position)

Illustration: The gravitational force is conservative

Consider a block that travels up an incline along the two paths indicated in the figures to the right. Is the work done by gravity path-independent or not?

Movement along incline

For movement along the incline, the work done by gravity is given by

$$\begin{aligned} W_g &= F_{g_{||}} L \\ &= -(mg\sin\theta)L \\ &= -mgL\sin\theta \end{aligned}$$

Remember, the (-) sign comes in here because gravity here is hindering the motion, not helping it.

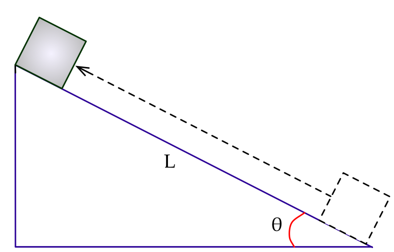

Figure 2: Path along incline

Lifting up, then over

We can find the work done by gravity for this path by dividing the path into two portions, lifting straight up, and then moving the block straight to the left.

$$W_{grav} = W_{up} + W_{over}$$

W_{up} is given by

$$\begin{aligned} W_{up} &= F_{g_{||}} d_{up} \\ &= -(mg)(L \sin\theta). \end{aligned}$$

Whereas W_{over} is given by

$$\begin{aligned} W_{over} &= F_{g_{||}} d_{over} \\ &= (0)(L \cos\theta) = 0 \end{aligned}$$

The work done in this portion is zero because the component of gravity parallel to the motion is zero - the direction of gravity is perpendicular to the displacement in the second portion of the path. Adding the two up, we get

$$\begin{aligned} W_{grav} &= W_{up} + W_{over} \\ &= -mgL \sin\theta. \end{aligned}$$

This is the same expression we got from the first path! This shows that the work done by gravity is path-independent, and so it is a conservative force.

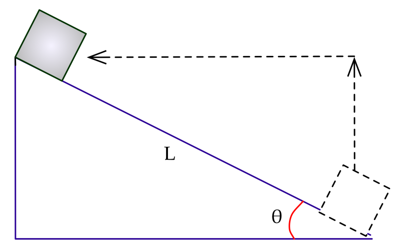

Figure 3: Lift up, then move over.

Potential Energy

POTENTIAL ENERGY is the energy stored in a system. Thus, it also tells us how much energy is available for the system to do work. If a system is conservative, then the work done in going from an initial state to a final state is given by the negative of the change in potential energy.

$$W_{i \to f} = -\Delta U = -(U_f - U_i) \quad (1)$$

We also know that for *all* systems, the work-energy theorem is true:

$$W_{i \to f} = \Delta K = K_f - K_i,$$

where K is the kinetic energy of the system. The above two equations imply that for a conservative system,

$$\Delta K = -\Delta U. \quad (2)$$

Gravitational potential energy near the Earth's surface

Near the earth's surface, the gravitational force can be approximated as $m\vec{g}$. Consider the motion of the ball in the figure to the right.

It travels in the vertical direction from initial position $\vec{y_i}$ to final position $\vec{y_f}$. The distance between the two points is simply $|\vec{y_f} - \vec{y_i}|$ which we can simply write as $y_f - y_i$ because we are considering one-dimensional motion. The work done by gravity for this process is given by

$$\begin{aligned} W_{i\to f} &= F_{g_{||}} d \\ &= -mg(y_f - y_i) \\ &= -(mgy_f - mgy_i) \end{aligned}$$

But from (1), we have $W_{i\to f} = -(U_f - U_i)$. Comparing these two equations, we get $U_f = mgy_f$, and $U_i = mgy_i$. We can generalize these results to say that the gravitational potential of an object near the Earth's surface is given by

$$U_{grav} = mgy$$

Here, y does not necessarily have to be the distance above the ground. We can choose our origin, that is, the $y = 0$ point according to our convenience.

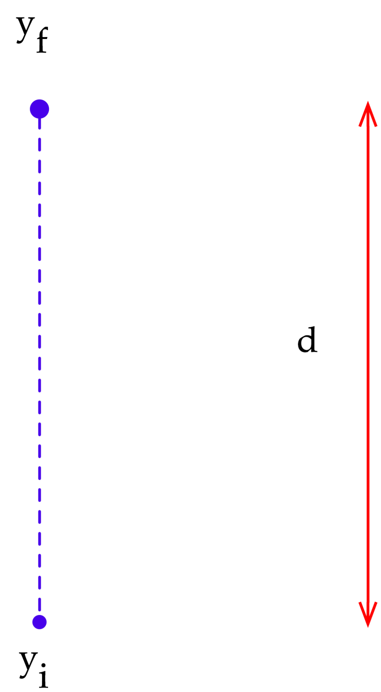

Figure 4: The object is moved upwards in one dimension from y_i to y_f, where $y_f - y_i = d$

Conservation of Mechanical Energy

In systems in which all the forces are conservative, we have from (2) that

$$\begin{aligned} \Delta K &= -\Delta U \\ \Rightarrow K_f - K_i &= -(U_f - U_i) \\ \Rightarrow K_f + U_f &= K_i + U_i \\ \Rightarrow ME_f &= ME_i \end{aligned}$$

Where ME denotes the mechanical energy of the system, given by the sum of the kinetic and potential energy.

$$ME = K + U$$

From the above relations, we can write

$$\begin{aligned} ME_f - ME_i &= 0 \\ \Rightarrow \Delta ME &= 0 \end{aligned}$$

This implies that the mechanical energy of the system is a constant for systems in which the forces are conservative, that is, the mechanical energy is *conserved*! In such systems, the kinetic energy and the potential energy might fluctuate individually, but their sum remains constant. That is, kinetic energy can transform into potential energy and vice-versa, but the total mechanical energy remains unchanged.

Generalization to include non-conservative forces

What if we throw in non-conservative forces into the mix, like kinetic friction? We can deal with them as follows. The net work, W_{net}, done in a process is given by

$$W_{net} = W_{NC} + W_{cons},$$

where W_{NC} is the work done by the non-conservative forces, and W_{cons} is the work done by the conservative forces. But we know from (1) that $W_{cons} = -\Delta U$ and from the work energy theorem that $W_{net} = \Delta K$. This gives us

$$\begin{aligned} \Delta K &= W_{NC} + (-\Delta U) \\ \Rightarrow \Delta K + \Delta U &= W_{NC} \\ \Rightarrow K_f - K_i + U_f - U_i &= W_{NC} \\ \Rightarrow K_i + U_i + W_{NC} &= K_f + U_f \end{aligned}$$

$$\Rightarrow \boxed{ME_i + W_{NC} = ME_f}$$

Example: Trapeze artist

A trapeze artist of mass m stands on an elevated platform. She takes hold of a horizontal rope of length L, steps off the platform and swings down in a circular arc. Find her speed when she reaches the bottom of the swing.

Start with the equation we just derived,

$$\begin{aligned} ME_i + W_{NC} &= ME_f \\ \Rightarrow K_i + U_i + W_{NC} &= K_f + U_f \\ \Rightarrow \frac{1}{2}mv_i^2 + mgy_i + W_{NC} &= \frac{1}{2}mv_f^2 + mgy_f \end{aligned}$$

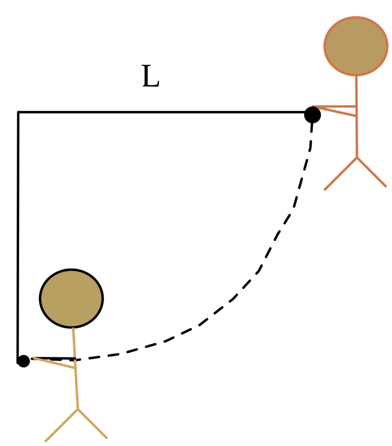

Figure 5: The dotted line indicates the trajectory of the trapeze artist. What is her velocity at the bottom of the swing?

For our convenience, we can take the $y = 0$ level to coincide with the bottom-most point of the swing. This gives us $y_i = L$ and $y_f = 0$. The artist also starts from rest, giving us $v_i = 0$. Tension does no work in this process, since it is always perpendicular to the direction of the displacement (or motion). This gives us $W_{NC} = 0$. Putting all of these facts together, we get

$$0 + mgL + 0 = \frac{1}{2}mv_f^2 + 0$$
$$\Rightarrow v_f = \sqrt{2gL}$$

Exercise: Compare this to the final speed of the artist if she just fell a distance L under free-fall conditions. Do you get the same answer? What does this tell you about the path-dependence (or lack thereof) of the work done by gravity?

Example: System with both conservative and non-conservative forces.

A block is released from rest and slides down a 3m long frictionless incline (at an angle of 20^o with the horizontal) and then over a rough horizontal floor with $\mu_k = 0.25$. How far from the bottom of the incline does the block come to rest?

Since there is friction in this system, mechanical energy is not conserved. If we take into account this non-conservative force, our general equation becomes

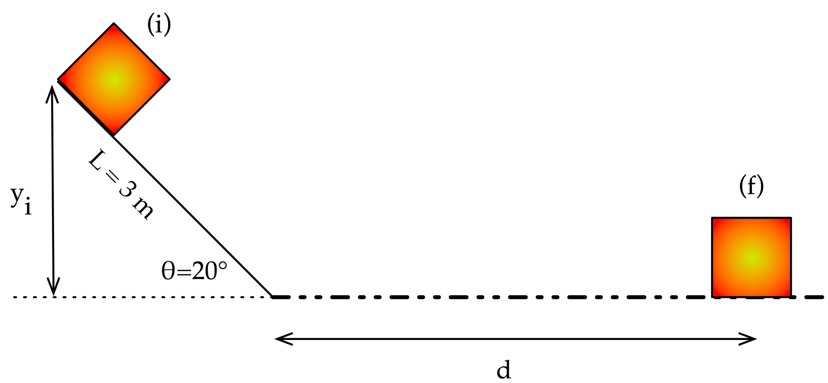

Figure 6: The block slides down a smooth incline and across a rough surface and comes to a stop.

$$K_i + U_i + W_{fric} = K_f + U_f \tag{3}$$
$$\Rightarrow \frac{1}{2}mv_i^2 + mgy_i + W_{fric} = \frac{1}{2}mv_f^2 + mgy_f \tag{4}$$

The initial and final speeds (v_i and v_f) are both zero, since the block starts from rest, and comes to a stop at the end. If we set our $y = 0$ level at the ground, we get $y_i = L\sin\theta$ and $y_f = 0$. The work done by friction is given by

$$W_{fric} = -f_k d$$
$$= -\mu_k N d$$

The normal force during the time that friction acts on the block can be found by applying Newton's second law to the y-direction. Doing this[2], we get $N = mg$. Putting these pieces back into equation (4), we get

[2] Work this out explicitly, or convince yourself that this is true! I will be skipping some steps in the notes in the interest of readability.

$$0 + mgL\sin\theta - \mu_k mgd = 0 + 0$$
$$\Rightarrow mgL\sin\theta = \mu_k mgd$$
$$\Rightarrow d = \frac{L\sin\theta}{\mu_k}$$

Substituting our numbers into the above expression, we get

$$\frac{L\sin\theta}{\mu_k} = \frac{3\sin 20^o}{0.25} = \boxed{4.1 \text{ metres}}$$