Supersymmetry in 0 dimensions

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Introduction

The standard model of particle physics provides a comprehensive framework that incorporates the fundamental particles that comprise the universe and the interactions between them. It also provides some of the most precise (and experimentally verified) predictions in the whole corpus of science, like the measurement of the fine structure constant. However, it is not the final word on the matter by any means.

It still has a large number of free parameters, more than can be comfortably reconciled with the label of a 'fundamental theory'. There is also the so-called *hierarchy problem* that troubles many theoretical particle physicists. The essence of the hierarchy problem is this: every time we have increased the energy with which we collide particles, we have discovered new, more massive, and more fundamental particles. The first example of discovery of previously unknown substructure of a fundamental particle was Rutherford's gold-foil scattering experiment. The experiment showed that atoms are not, in fact indivisible as previously thought, but are mostly empty space with a hard nucleus at the center. And more recently of course, we have seen that the nucleus is made of protons and neutrons, which are in turn made up of quarks held together by gluon-mediated forces. We have also seen the discovery of heavier fundamental particles like the top quark (which has a rest mass larger than that of the proton) as we raised the energy threshhold. Based on this trend, it is natural to expect that we will see at least *some* new physics at higher energies[1].

[1] Indeed, it is widely accepted that the standard model is simply a low-energy approximation to some more fundamental underlying theory.

The Higgs boson, discovered recently at the Large Hadron Collider has a experimentally measured mass of 126 GeV/c^2. In the standard model, this mass is a free parameter. However, in almost any extension of the standard model that introduces new, heavier (and therefore as-yet undiscovered) particles, the Higgs mass suffers from extremely large quantum corrections (approximately 30 orders of magnitude larger!) due to couplings to these particles. So the question is, why is the mass of the Higgs boson the value it is and not much higher? Supersymmetry offers an attractive solution to this problem. It turns out that a quantum correction from a fermion can cancel out the quantum correction from a boson and vice versa. Supersymmetry predicts that there is a symmetry between bosonic and fermionic states, and there is a transformation that takes one state to the other. The result is a host of new particles, so called *superpartners* to the familiar standard model particles, that provide the precise cancellations required to keep the Higgs mass from diverging. The elegant structure of supersymmetry has attracted a lot of attention from both mathematicians and physicists. At the Large Hadron Collider, physicists are looking for evidence for or against it. The search is still ongoing. Although supersymmetry is mainly interesting in the context of high-energy physics, we can also look at its mathematical structure in 0-space dimensions, a viewpoint introduced by Ed Witten in 1982 [5] and use it to obtain a well-known result in introductory quantum mechanics, the spectrum of a quantum harmonic oscillator.

Supersymmetric operators in 0-dimensions

Let us consider a Hilbert space[2] $\mathcal{H}$, self-adjoint operators[3] H and Q, and a bounded[4] self-adjoint operator P in $\mathcal{H}$, such that the following conditions are satisfied:

$$H = Q^2 > 0 \tag{1}$$
$$P^2 = 1 \tag{2}$$
$$\{Q, P\} := QP + PQ = 0. \tag{3}$$

We then say that the system $\{H, P, Q\}$ has *supersymmetry*. This is just a definition. Now, since P is self-adjoint, it has real eigenvalues (See Dimock, p. 16). Consider $\psi \in \mathcal{H}$, an eigenvector of P with eigenvalue λ. Now, $P\psi = \lambda\psi$, so $P^2\psi = \lambda^2\psi$. But from (2), we know that $\lambda^2 = 1$. And since λ has to be real, it can only take on one of two values, +1 and -1. Let us denote the two collections of eigenvectors of P in $\mathcal{H}$ that correspond to these two eigenvalues as $\mathcal{H}_f$ and $\mathcal{H}_b$. In set-builder notation,

$$\mathcal{H}_f := \{\psi \in \mathcal{H} | P\psi = -\psi\} \tag{4}$$
$$\mathcal{H}_b := \{\psi \in \mathcal{H} | P\psi = \psi\} \tag{5}$$

The reader with a physics background might correctly guess that the f and b denote fermionic and bosonic subspaces respectively. We can then uniquely decompose $\mathcal{H}$ into the *direct sum*[5] of the vector spaces $\mathcal{H}_f$ and $\mathcal{H}_b$, that is, we can write

$$\mathcal{H} = \mathcal{H}_f \oplus \mathcal{H}_b$$

Thus we can write

[2] A **Hilbert space** $\mathcal{H}$ is a *complete* normed complex vector space in which the *norm* comes from an *inner product*.

The **norm** is a real-valued function on $\mathcal{H}$, sending $f \in \mathcal{H}$ to $\|f\|$, which satisfies

- $\|cf\| = |c|\|f\|$ for $c \in \mathbb{C}$.
- $\|f + g\| \leq \|f\| + \|g\|$
- $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$.

$\mathcal{H}$ is **complete** if every Cauchy sequence in it has a limit that is also in it. That is, if $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, then there exists a unique $f \in \mathcal{H}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

The **inner product** on $\mathcal{H}$ is a map from pairs $f, g \in \mathcal{H}$ to $(f, g) \in \mathbb{C}$ such that

- (f, g) is linear in f & anti-linear in g.
- $\overline{(f, g)} = (g, f)$
- $(f, f) \geq 0$ and $(f, f) = 0$ iff $f = 0$

(Physicists might be more familiar with the Dirac notation for the inner product, $\langle f|g\rangle$)

[3] An operator T is **self-adjoint** if it is equal to its *adjoint*, T^*, defined by the inner product relation

$$(T^*g, f) = (g, Tf).$$

[4] A linear operator T is said to be **bounded** if there is a constant M such that $\|Tf\| \leq M\|f\|$ for all $f \in \mathcal{H}$.

[5] $\mathcal{H}$ is the **direct sum** of $\mathcal{H}_f$ and $\mathcal{H}_b$ if

- $\mathcal{H}_f$ and $\mathcal{H}_b$ have no common members.
- Every element of $\mathcal{H}$ can be written as a sum of a member of $\mathcal{H}_b$ and a member of $\mathcal{H}_b$.

We know that there are no common members because the members of $\mathcal{H}_f$ and $\mathcal{H}_b$ have distinct eigenvalues - it would be impossible for a member to have both -1 and +1 as eigenvalues simultaneously. The second condition is satisfied as well, because by the *spectral theorem*, since P is a bounded self-adjoint operator on $\mathcal{H}$, all the vectors in $\mathcal{H}$ are linear combinations of eigenvectors of P.

$$P = \begin{pmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{pmatrix}$$

where $\mathbb{1}_b$ and $\mathbb{1}_f$ are the unit operators acting on $\mathcal{H}_b$ and $\mathcal{H}_f$ respectively. Basically, we can think of vectors in the Hilbert space $\mathcal{H}$ as having fermionic and bosonic components. The operator P will act differently on the different components. Bosonic states are invariant under P (which is to be interpreted as a parity inversion operation, inverting the signs of the coordinates in the wave function), while fermionic states pick up a negative sign. The dimensions of the subspaces will depend upon the physical system in question, that is, the number of particles, the number of degrees of freedom, &c.

We can find the form of the operator Q (the matrix representation) by considering the following. P and Q anti-commute. Let Q take the form

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We will determine the elements a, b, c, d. Since P & Q anti-commute, we have

$$\begin{aligned} \{P, Q\} = PQ + QP &= \begin{pmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbb{1}_b & 0 \\ 0 & -\mathbb{1}_f \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -c & -d \end{pmatrix} + \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & -2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &\Rightarrow a = d = 0 \end{aligned}$$

And since we know that Q is self-adjoint, we have $Q^* = Q$, i.e.

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c^* \\ b^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \tag{6}$$

which implies b and c are adjoints of each other. Thus we can write Q in the form

$$Q = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

We can think of A^* and A as the familiar creation and annihilation operators. We will see this more explicitly when we come to the harmonic oscillator spectrum later. But first we will prove a theorem that will help us later on.

Theorem: If the system $\{H, P, Q\}$ has supersymmetry, then for any bounded open set $\Omega \subseteq (0, \infty)$ we have[6]

$$\dim(E_\Omega(H) \upharpoonright \mathcal{H}_b) = \dim(E_\Omega(H) \upharpoonright \mathcal{H}_b)$$

where $E_\Omega(H)$ is the spectral projector[7] of H on Ω.

[6] For an operator A defined on a vector space W, $A \upharpoonright V$ is the *restriction* of A to V, a subset of W.

[7] Spectral projector: This operator acts on members $f \in \mathcal{H}$ and returns 1 if f is an eigenvector of the operator H with an eigenvalue that lies in the set Ω, and 0 otherwise.

Proof: Basically, the quantity $\dim(E_\Omega(H) \upharpoonright \mathcal{H}_b)$ is the number of bosonic states that have eigenvalues that lie in the set Ω. This theorem says that there are an equal number of bosonic and fermionic eigenstates with eigenvalues that lie in Ω. This is the one-to-one correspondence between bosons and fermions that supersymmetry gives us.

Let $P_\pm$ be the projectors onto the subspaces $\mathcal{H}_b$ and $\mathcal{H}_f$ respectively. Let us also define $E_\Omega^\pm = E_\Omega(H)P_\pm$. Our immediate goal is now to show that

$$QE_\Omega^\pm = E_\Omega^\mp Q, \tag{7}$$

$$\text{i.e. } QE_\Omega(H)P_\pm = E_\Omega(H)P_\mp Q. \tag{8}$$

It may not be the most elegant method, but to me the most straightforward way to proceed is to use explicit matrix representations of these operators, assuming they act on a vector $\psi \in \mathcal{H}$ comprised of bosonic and fermionic components.

$$\psi = \begin{pmatrix} \psi_b \\ \psi_f \end{pmatrix}$$

The matrix representations of the operators acting on this vector are:

$$P_+ = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}; \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \text{ and } E_\Omega(H) = \begin{pmatrix} E_\Omega(H)_b & 0 \\ 0 & E_\Omega(H)_f \end{pmatrix}$$

Combining these with the representation of Q that we have already seen, it is fairly easy to see that $P_\pm$ commutes with $E_\Omega(H)$. So if we can show that $QP_\pm = P_\mp Q$, then we should be able to get (8). Let us show this explicitly for one of the cases.

$$QP_+\psi = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_b \\ \psi_f \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi_b \end{pmatrix}$$

$$P_-Q\psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_b \\ \psi_f \end{pmatrix} = \begin{pmatrix} 0 \\ A\psi_b \end{pmatrix}$$

Similarly, we can show that $QP_- = P_+Q$, and so we finally have $QE_\Omega^\pm = E_\Omega^\mp Q$.

Now, since $0 \notin \Omega$, Q is invertible, and

$$\dim(E_\Omega^+) = \dim(E_\Omega^-)$$

But $\dim(E_\Omega(H) \restriction \mathcal{H}_b) = \dim(E_\Omega^+)$ and $\dim(E_\Omega(H) \restriction \mathcal{H}_f) = \dim(E_\Omega^-)$, so the theorem is proved.

Spectrum of a harmonic oscillator using supersymmetric operators

Let the Hilbert space $\mathcal{H} = L^2(\mathbb{R} \otimes \mathbb{C}^2)$ and let $q(x)$ be polynomial function of x. If we define the 'creation' and 'annihilation' operators

$$A^* = -\frac{d}{dx} + q(x)$$
$$A = \frac{d}{dx} + q(x)$$

Then we can find the products A^*A and AA^* as follows. Let us first consider the product A^*A, acting on $f \in \mathcal{H}$.

$$\begin{aligned}
A^*Af &= \left(-\frac{d}{dx} + q(x)\right)\left(\frac{d}{dx} + q(x)\right) f \\
&= \left(-\frac{d^2}{dx^2} + q^2(x)\right) f + q(x)\frac{df}{dx} - \frac{d}{dx}\left(q(x)f\right) \\
&= \left(-\frac{d^2}{dx^2} + q^2(x)\right) f + q(x)\frac{df}{dx} - \left(\frac{dq(x)}{dx}\right) f - q(x)\frac{df}{dx} \\
&= \left(-\frac{d^2}{dx^2} + q^2(x) - q'(x)\right) f
\end{aligned}$$

We can perform a similar operation for AA^*, and obtain

$$AA^* = -\frac{d^2}{dx^2} + q^2(x) - q'(x).$$

Then we can write the operator Q discussed earlier as

$$Q = \begin{pmatrix} 0 & -\frac{d}{dx} + q(x) \\ \frac{d}{dx} + q(x) & 0 \end{pmatrix}$$

So then the operator $H = Q^2$ becomes

$$H = Q^2 = \begin{pmatrix} -\frac{d^2}{dx^2} + q^2(x) - q'(x) & 0 \\ 0 & \frac{d^2}{dx^2} + q^2(x) + q'(x) \end{pmatrix}$$

So the system $\{H, P, Q\}$ is supersymmetric. If we set $q(x) = x$, then

$$A^*A = -\frac{d^2}{dx^2} + q^2(x) - 1$$

which is nothing but the harmonic oscillator, shifted by 1. Similarly,

$$AA^* = -\frac{d^2}{dx^2} + q^2(x) + 1,$$

implying that

$$AA^* = A^*A + 2. \tag{9}$$

The above line implies

$$\dim(E_\Delta(A^*A)) = \dim(E_{\Delta-2}(AA^*)) \text{ for } \Delta \subseteq (0,\infty) \tag{10}$$

That is, for every eigenstate of A^*A with eigenvalue in some set Δ, there will be an eigenstate of AA^* with eigenvalue in a set negatively shifted from Δ by two. For example, if there are 25 eigenstates of A^*A with eigenvalues in the set (3,5), there will be 25 eigenstates of AA^* with eigenvalues in the set (1,3). We know that the spectrum of H is the union of the spectrum of A^*A and the spectrum of AA^*, i.e.

$$\sigma(H) = \sigma(A^*A) \cup \sigma(AA^*).$$

Since $A^*A \geq 0$, and using (9), we can say that $AA^* \geq 2$, so

$$\sigma(AA^*) \geq 2,$$

i.e. the eigenvalues of AA^* must be greater than or equal to 2. Now, from (10), we can say that since there are no eigenstates of AA^* with eigenvalues in (0,2), there are no eigenstates of of A^*A with eigenvalues in (2,4). We can recursively apply this to say that in general, there is no spectrum of H in $(2n, 2(n+1))$, where n is a natural number. Therefore, H must have spectrum in the set $\{2, 4, 6, 8, ...\}$. Save for some physical constants and scaling factors, we have recovered the essential discrete nature of the quantum harmonic oscillator spectrum using a novel approach. This is just one of the ways that the mathematics of supersymmetry can cast new light upon familiar physics.

Bibliography

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