Moment of Inertia and Torque.

1. Demo: Inertia rods → It is harder to rotate the rods that have mass distributed further from the axis.

2. Moment of Inertia

   ⇒ Rotational analogue of mass.

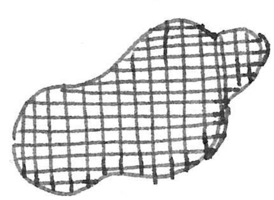
   - Higher mass ⇒ harder to move
   - Higher moment of inertia ⇒ harder to rotate.

If an object is comprised of point masses $m_1, m_2, \ldots m_n$ at distances $r_1, r_2, \ldots, r_n$ from the axis of rotation, the moment of inertia, I, is given by

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \boxed{\sum_n m_n r_n^2}$$

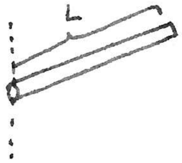
The units of I are ~~kg m/s~~ $kg\, m^2$

Of course, in real life, objects are continuous bodies, and their moments of inertia are calculated using ~~calcula~~ calculus (You can approximate the continuous object as a collection of point masses)



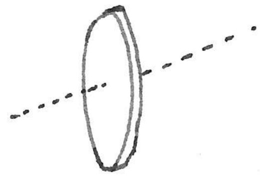
↑
calculus in action!
(kind of.)

Some common moment-of-inertia examples.



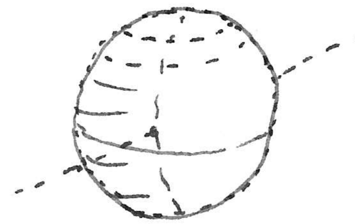
Rod of mass M,
Axis at one end,
perpendicular to it.

$$\boxed{I = \frac{1}{3} M L^2}$$



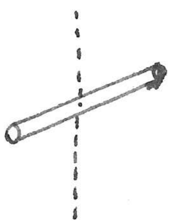
Disc or cylinder
of mass M,
radius R, axis
through center,
perpendicular to plane

$$\boxed{I = \frac{MR^2}{2}}$$



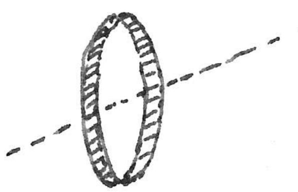
Solid sphere,
axis through
center, radius R,
mass M

$$\boxed{I = \frac{2}{5} M R^2}$$



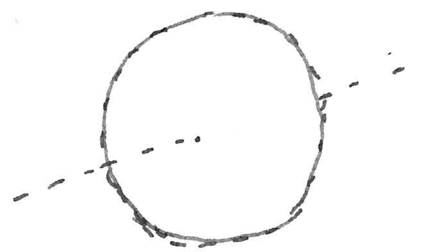
Axis through
center of rod.

$$\boxed{I = \frac{1}{12} M L^2}$$



Hoop or tube,
axis through
center, $\perp$ to plane

$$\boxed{I = M R^2}$$



Hollow sphere

$$\boxed{I = \frac{2}{3} M R^2}$$

Torque

→ Rotational analogue of force.

↳ More torque ⇒ greater angular acceleration

Consider ~~of~~ opening a door.

① The more force you apply, the faster the door rotates (opens)

⇒ Torque depends on <u>magnitude</u> of applied force.

② If the same amount of force is applied, where do you push the door to make it easy?

(a) At the hinge

(b) In the middle

(c) At the handle (edge furthest from the axis)

⇒ Torque depends on <u>location</u> of application of force.

③ What direction should the force be applied to make the opening easier?

(a) Perpendicular to the door frame

(b) Parallel to the door

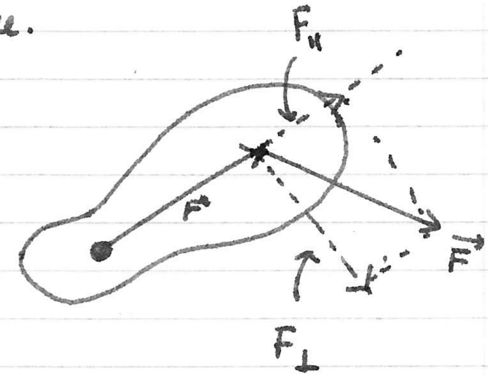
(c) Some other angle

⇒ Torque depends on the direction of the applied force.

Mathematical definition of torque.

The magnitude of the ~~torq~~ torque, τ, is given by

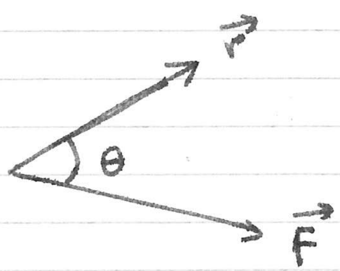
$\boxed{|\vec{\tau}| = rF_{\perp}}$ where r is the distance of the force application point from the axis,



and $F_{\perp}$ is the component of force perpendicular to the line on which $\vec{r}$ lies.

Units of torque: N m.

$$\boxed{|\vec{\tau}| = rF_{\perp} = rF\sin\theta},$$ where θ is the angle between $\vec{r}$ & $\vec{F}$.



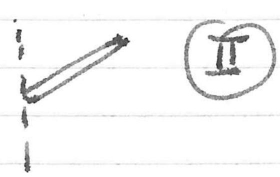
τ is largest when $\theta = 90^\circ$

τ is smallest when $\theta = 0$ or $\theta = 180^\circ$

Example: Moment of inertia ~~of a~~ and torque.

Consider two rods, (identical), rotating about different axes, with the same torque applied to each.

$$I = \frac{ML^2}{12}$$



$$I = \frac{ML^2}{3}$$

Which one gets a larger angular acceleration?

→ Rod I, since it has a smaller $I \Rightarrow$ less resistance to rotation. But how much larger??

Rotational Dynamics.

1. Analogue of Newton's 2^{nd} Law for rotation.

Translation : $\sum \vec{F}_i = m\vec{a}$

Rotation : $\boxed{\sum \vec{\tau} = I \vec{\alpha}}$

net torque — moment of inertia — angular acceleration

2. General strategy for rotational dynamics.

(i) Draw FBD showing forces applied at correct locations (mg acts at center of symmetric bodies)

(ii) Find torques due to all the forces about the axis of rotation.

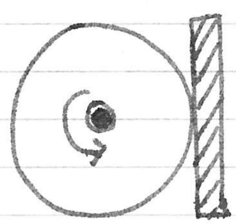
Sign convention: (+) for CCW torques
(−) for CW torques.

(iii) $\sum \tau = I\alpha \longrightarrow$ solve for α.

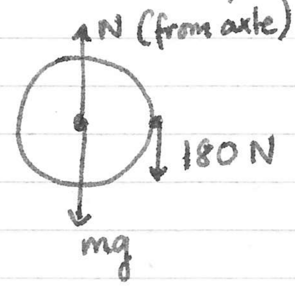
(iv) Once (constant) α is found, use rotational kinematics equations to find ω and $\Delta\theta$.

Example: Braking of a wheel.

In a certain machine, a rotating wheel is pressed against a rubber brake pad in order to stop the wheel's rotation. The wheel is a disc of mass 32 kg and radius 1.2 m, initially rotating at 250 rpm. The brake pad exerts a constant 180 N force in the direction opposite to the wheel's rotation. Find the time required for the wheel to stop.



FBD:



$$\Sigma \tau = I\alpha$$

$$\Rightarrow -rF\sin\theta = \frac{MR^2}{2}\alpha \qquad \left[I = \frac{MR^2}{2} \text{ for a solid disk} \right]$$

$$\Rightarrow \alpha = -\frac{2RF}{MR^2}$$

[$\theta = 90°$ here, $r = R$, since force is applied tangent to disk]

$$\Rightarrow \alpha = -\frac{2F}{MR}$$

[$F = 180\,\mathrm{N}$, $R = 1.2\,\mathrm{m}$, $M = 32\,\mathrm{kg}$]

[(−) sign since τ is CW]

Now, rotational kinematics.

Some forces (F) cause torque
Some forces (N, mg) do not!

$$\omega_f = \omega_i + \alpha t$$

$$\cancel{\Rightarrow 0 = 250 - \frac{2F}{MR}t}$$

$$\cancel{\Rightarrow t = (250)\frac{MR}{2F} = (250)(32)\frac{(1.2)}{2(180)}}$$

$$\cancel{= \boxed{2.79\,\mathrm{s}}}$$

$$\Rightarrow t = \frac{\omega_f - \omega_i}{\alpha}, \qquad \omega_i = +250\,\mathrm{rpm} = \frac{250(2\pi)\,\mathrm{rad}}{60\,\mathrm{s}}$$

$$\omega_f = 0\ \mathrm{rad/s}$$

$$\Rightarrow t = \frac{0 - \frac{250(2\pi)}{60}}{\frac{-2(180)}{32(1.2)}} = \boxed{2.79\,\mathrm{s}}$$

Rotational Kinetic Energy

Translation: $K_{trans} = \frac{1}{2} m v^2$

$$K_{rot} = \frac{1}{2} I \omega^2$$

K of the body

$$= \sum_i \frac{1}{2} m_i v_i^2$$

$$= \sum_i \frac{1}{2} m_i (r_i \omega)^2$$

$$= \frac{1}{2} \sum_i \omega^2 (m_i r_i^2)$$

$$= \frac{1}{2} \omega^2 \left(\sum_i m_i r_i^2 \right)$$

$$= \frac{1}{2} \omega^2 I$$

$$= \boxed{\frac{1}{2} I \omega^2}$$