Simple Harmonic Motion

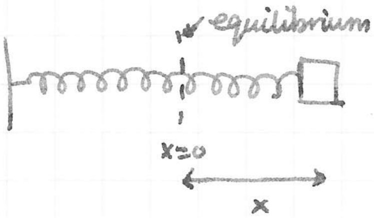
Remember when we discussed why the study of springs was worthwhile?

No?

Well, just remember this then: many ~~realt~~ real-world systems can be approximately modeled as systems of springs.

The kind of motion such systems engage in is called simple harmonic motion.

Consider a mass on a spring. If we stretch/compress it initially and then let it go, it oscillates back and forth. ~~tan~~ How can we describe its motion?



What ~~is~~ are $x(t)$, $v(t)$, $a(t)$?

~~$\Sigma F/x$~~ From Newton's second law,

$$\sum F_x = ma_x$$
$$\Rightarrow -kx = ma$$
$$\Rightarrow a = -\frac{k}{m}x$$

$$\Rightarrow a(t) = -\frac{k}{m}x(t) \Rightarrow \text{it is not constant.}$$

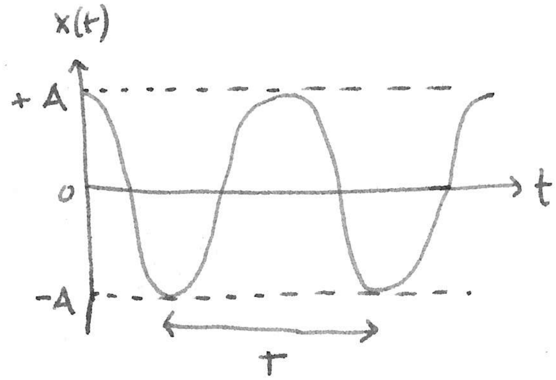
$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Solving this requires calculus.

What does the motion depend on?
(i) spring constant
(ii) Max stretching/compression
(iii) Attached mass.

Spring stretched and released from rest

Position



$$\boxed{x(t) = x_{max} \cos(\omega t)}$$

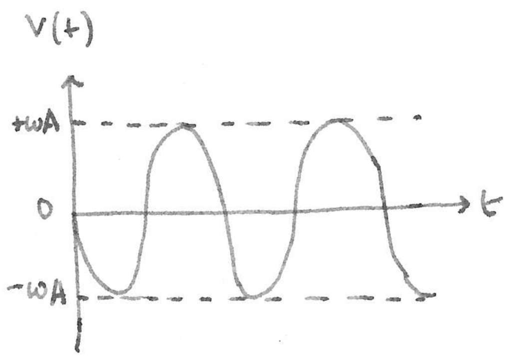
$x_{max} = A$ (amplitude)

$\omega = \sqrt{\frac{k}{m}}$ → Angular frequency (rad/s)

$T = \frac{2\pi}{\omega}$ → Time period (s)

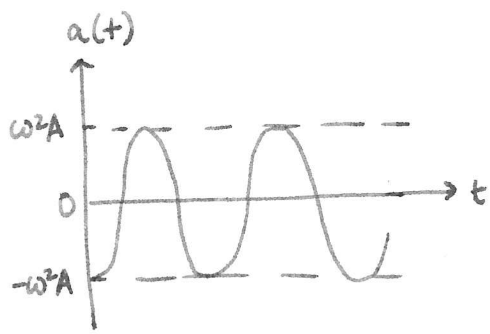
~~⇒ ω = 2~~

$f = \frac{1}{T}$ → frequency. ($^1/_s$ or Hz)



$$\boxed{V(t) = -V_{max} \sin(\omega t)}$$

$V_{max} = \omega A$



$$\boxed{a(t) = -a_{max} \cos(\omega t)}$$

$a_{max} = \omega^2 A$

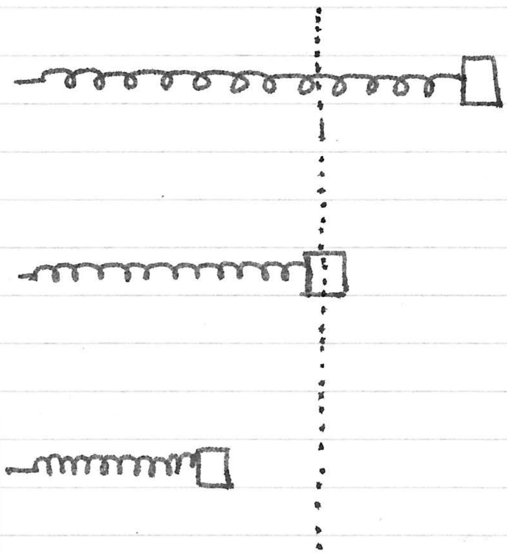
Q: How did we get $a_{max} = \omega^2 A$?

$\Sigma F = ma \Rightarrow -kx = ma \Rightarrow a = -\frac{k}{m} x$

$\Rightarrow |a| = \frac{k}{m} |x|$

$\Rightarrow |a_{max}| = \frac{k}{m} |x_{max}| = \omega^2 |x_{max}| = \omega^2 A$

Energy over one cycle.



U	K	ME = K + U
U_{max}	0	$\frac{1}{2}kA^2$
0	K_{max}	$\frac{1}{2}mV_{max}^2$
U_{max}	0	$\frac{1}{2}kA^2$

Since the spring force is conservative, mechanical energy is conserved.

$$\Rightarrow \frac{1}{2}kA^2 = \frac{1}{2}mV_{max}^2$$

$$\Rightarrow V_{max} = \sqrt{\frac{k}{m}}\,A = \omega A.$$

Energy as a function of time.

→ Easy, just substitute the values of $x(t)$ and $v(t)$ into the energy expressions.

$$U(t) = \frac{1}{2}k[\cancel{k}\, x(t)]^2 \qquad K(t) = \frac{1}{2}m[v(t)]^2$$

$$= \frac{1}{2}k[A\cos(\omega t)]^2 \qquad = \frac{1}{2}m[-\omega A\sin(\omega t)]^2$$

$$= \boxed{\frac{1}{2}kA^2\cos^2(\omega t)} \qquad = \frac{1}{2}m\omega^2A^2\sin^2(\omega t) = \boxed{\frac{1}{2}kA^2\sin^2(\omega t)}$$

$$ME(t) = K(t) + U(t) = \frac{1}{2}kA^2\left[\sin^2(\omega t) + \cos^2(\omega t)\right] = \boxed{\frac{1}{2}kA^2}$$ → Constant (as expected)

FOD FRI?

Example: Vibration Test Apparatus

Mechanical devices are often tested for vibration resistance in an apparatus that simulates conditions of daily use. Suppose that a small device of mass 0.35 kg is required to withstand 100 oscillations in 60 s at a maximum acceleration of $2g$. Assuming that friction can be neglected, find (a) the stiffness of the spring to which this device must be attached, (b) the amplitude of the motion, and (c) the maximum speed of the device during the test.

DO

Given: $m = 0.35$ kg ; $T = (60\text{ s})/(100) = 0.6$ s ; $a_{max} = 2g = 19.6\text{ m/s}^2$

Known equations: $\omega = \sqrt{k/m}$; $T = 2\pi/\omega$; $a_{\max} = \omega^2 A$; $v_{\max} = \omega A$

Write

(a) Solve for ω from T, then k from ω

$$0.6 = T = 2\pi/\omega$$

$$\Rightarrow \omega = 10.47 \text{ rad/s}$$

$$\omega = \sqrt{k/m}$$

$$10.47 = \sqrt{k/(0.35)}$$

$$\Rightarrow \underline{\underline{k = 38.4 \text{ N/m}}}$$

(b) Solve for A from known a_{max}

$$a_{\max} = \omega^2 A$$

$$19.6 = (10.47)^2 A$$

$$\underline{\underline{A = 0.179 \text{ m}}}$$

(c) Calculate v_{max}

$$v_{\max} = \omega A$$

$$= (10.47)(0.179)$$

$$= \underline{\underline{1.87 \text{ m/s}}}$$